On generalized b*-Closed Sets In Topological Spaces

Zinah T. ALhawez

Dept. of Mathematics/College of Education for Woman / University of Tikrit

Received in:22/December/2014 , Accepted in:20/September/2015

Abstract
In this paper, we introduce and study the concept of a new class of generalized closed set which is called generalized b*-closed set in topological spaces (briefly .g b*-closed) we study also, some of its basic properties and investigate the relations between the associated topology.

Keywords: gb* -closed set, gb -closed set,g-closed set.
Introduction

Levine[9] introduced the concept of generalized closed sets (briefly, g-closed) and studied their most fundamental properties in topological spaces. Arya and Nour[6], Bhattacharya and Lahiri[7], Levine[10], Mashhour[11], Njastad[13] and Andrijevic[3,4] introduced and investigated generalized semi-open sets, semi generalized open sets, open sets, semi-open sets, pre-open sets and α-open sets, semi pre-open sets and b-open sets which are some of the weak forms of open sets and the complements of these sets are called the same types of closed sets. A.A.Omari and M.S.M.Noorani[14] introduced and studied the concept of generalized b-closed sets (briefly gb-closed) in topological spaces. Recently Sundaram and Sheik John [15] introduced and studied w-closed sets. S.Muthuvel and R.Parimalazhagan [12] introduced and studied b*closed sets, A.Poongothai and R.Parimalazhagan [5] introduced and studied strongly b*-closed set in topological spaces.

In this paper, we introduce a new class of sets, namely gb*-closed sets for topological spaces. This class lies between the class b*-closed set and strongly b*-closed set.

2. Preliminaries

Let (X,T) be topological spaces and A be a subset of X. The closure of A and interior of A are denoted by \(cl(A)\) and \(int(A)\) respectively, union of all b-open (semi-open, pre-open, α-open) sets \(X\) contained in \(A\) is called b-interior (semi-interior, pre-interior, α-interior, respectively) of \(A\), it is denoted by \(b-int(A)\) (s-int(A), p-int(A), α-int(A), respectively). The intersection of all b-closed (semi-closed, pre-closed, α-closed) sets \(X\) containing \(A\) is called b-closure (semi-closure, pre-closure, α-closure, respectively) of \(A\) and it is denoted by \(bcl(A)\) (scl(A), pcl(A), acl(A), respectively). In this section, we recall some definitions of open sets in topological spaces.

**Definition 2-1[15]:** A subset \(A\) of a topological space \((X,T)\) is called a pre-open set if \(A \subseteq int(cl(A))\) and pre-closed set if \(cl(int(A)) \subseteq A\).

**Definition 2-2[10]:** A subset \(A\) of a topological space \((X,T)\) is called a semi-open set if \(A \subseteq cl(int(A))\) and semi-closed set if \(int(cl(A)) \subseteq A\).

**Definition 2-3[3]:** A subset \(A\) of a topological space \((X,T)\) is called a \(α\)-open set if \(A \subseteq int(cl(int(A)))\) and \(α\)-closed set if \(cl(int(cl(A))) \subseteq A\).

**Definition 2-4[8]:** A subset \(A\) of a topological space \((X,T)\) is called a \(β\)-open set if \(A \subseteq cl(int(cl(A)))\) and \(β\)-closed set if \(int(cl(cl(A))) \subseteq A\).

**Definition 2-5[1]:** A subset \(A\) of a topological space \((X,T)\) is called a \(b\)-open set if \(A \subseteq cl(int(A)) \cup int(cl(A))\) and \(b\)-closed set if \(int(cl(A)) \cap cl(int(A)) \subseteq A\).

**Definition 2-6[9]:** A subset \(A\) of a topological space \((X,T)\) is called a generalized –closed set (briefly, \(g\)-closed) if \(cl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is open set.

**Definition 2-7[7]:** A subset \(A\) of a topological space \((X,T)\) is called a semi generalized closed set (briefly, \(sg\)-closed) if \(scl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is semi-open set.
**Definition 2-8[8]:** A subset $A$ of a topological space $(X,T)$ is called a \textit{generalized $\alpha$-closed set} (briefly $g\alpha$-closed) if $\alphacl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\alpha$-open set.

**Definition 2-9[2]:** A subset $A$ of a topological space $(X,T)$ is called a \textit{generalized $b$-closed set} (briefly $gb$-closed) if $bcl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open set.

**Definition 2-10[8]:** A subset $A$ of a topological space $(X,T)$ is called a \textit{generalized $\beta$-closed set} (briefly $g\beta$-closed) if $\betacl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open set.

**Definition 2-11[5]:** A subset $A$ of a topological space $(X,T)$ is called \textit{weakly generalized closed set} (briefly $wg$-closed) if $cl(int(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is open set.

**Definition 2-12[15]:** A subset $A$ of a topological space $(X,T)$ is called \textit{weakly-closed set} (briefly $w$-closed) if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is semi-open set.

**Definition 2-13[12]:** A subset $A$ of a topological space $(X,T)$ is called \textit{generalized $b^*$-closed set} if $int(cl(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is $b^*$-open set.

**Definition 2-14[5]:** A subset $A$ of a topological space $(X,T)$ is called \textit{generalized $g^*$-closed set} if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $g^*$-open set.

**Definition 2-15[5]:** A subset $A$ of a topological space $(X,T)$ is called \textit{generalized $gb^*$-closed set} if $bcl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $gb^*$-open set.

**Definition 2-16[5]:** A subset $A$ of a topological space $(X,T)$ is called \textit{strongly $b^*$-closed set} (briefly, $sb^*$-closed) if $cl(int(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is $b^*$-open set.

**Definition 2-17[5]:** A subset $A$ of a topological space $(X,T)$ is called \textit{$b^**$-open set} if $A \subseteq int(cl(int(A))) \cup cl(int(cl(A)))$ and \textit{$b^**$-closed set} if $cl(int(cl(A)) \cap int(cl(int(A)) \subseteq A)$.

### 3. Generalized $b^*$-closed sets.

In this section, we introduce and study the concept of generalized $b^*$-closed set in topological spaces. Also we study the relationship between this set and the other types of sets.

**Definition 3-1:** A subset $A$ of a topological space $(X,T)$ is called \textit{generalized $b^*$-closed set} (briefly, $gb^*$-closed) if $int(cl(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is $gb^*$-open set.

**Theorem 3-2:** Every closed set is $gb^*$-closed set.

**Proof:** Assume that $A$ is a closed set in $X$ then $cl(A) = A$, and $U$ be any $gb^*$-open set where $A \subseteq U$. Since $int(A) \subseteq A$. implies that $int(cl(A)) \subseteq U$. Hence $A$ is $gb^*$-closed set in $X$. 

---

**Mathematics**

206 |
Remark 3-3: The converse of the Theorem [3-2] need not be true as seen by the following example.

Example 3-4: Let $X=\{a,b,c\}$ with $T=\{X, \emptyset, \{a\}\}$. In this topological space, the subset $A=\{b\}$ is gb*-closed set but not closed set.

Theorem 3-5: A set $A$ is gb*-closed set iff $\text{int}(\text{cl}(A))-A$ contains no non-empty gb-closed set.

Proof: Necessity: Suppose that $F$ is a non-empty gb-closed subset of $\text{int}(\text{cl}(A))$ such that $F \subseteq \text{int}(\text{cl}(A))-A$. Then $F \subseteq \text{int}(\text{cl}(A)) \cap A^c$. Since $F^c$ is gb-closed and $A$ is gb*-closed, $\text{int}(\text{cl}(A)) \subseteq F^c$. Therefore $F \subseteq (\text{int}(\text{cl}(A)))^c$. Thus $F \subseteq (\text{int}(\text{cl}(A)) - A)^c$. Therefore $F=\emptyset$ and this implies that $\text{int}(\text{cl}(A))-A$ contains no non-empty gb-closed set.

Sufficiency: Assume that $\text{int}(\text{cl}(A))-A$ contains no non-empty gb-closed. Let $A \subseteq U$, $U$ is gb-open set. Suppose that $\text{int}(\text{cl}(A))$ is not contained in $U$, then $\text{int}(\text{cl}(A)) \cap U^c$ is a non-empty gb-closed set of $\text{int}(\text{cl}(A))-A$ which is a contradiction. Therefore $\text{int}(\text{cl}(A))U \subseteq$ and hence $A$ is gb*-closed set.

Theorem 3-6: Let $B \subseteq Y \subseteq X$, if $B$ is gb*-closed set relative to $Y$ and that $Y$ is both gb-open and gb*-closed set in $(X,T)$ then $B$ is gb*-closed set in $(X,T)$.

Proof: Let $U \subseteq B$ and $U$ be a gb-open set in $(X,T)$. But Given that $B \subseteq Y \subseteq X$. Therefore $B \subseteq Y$ and $U \subseteq B$. This implies that $Y \cap U \subseteq B$. Since $B$ is gb*-closed set relative to $Y$, then $Y \cap U \subseteq \text{int}(\text{cl}(Y))$ (i.e) $Y \cap U \subseteq Y \cap \text{int}(\text{cl}(Y))$.implying that $U \subseteq Y \cap \text{int}(\text{cl}(Y))$.

Thus $U \cup [\text{int}(\text{cl}(B))]^c \subseteq [Y \cap \text{int}(\text{cl}(B))] \cup [\text{int}(\text{cl}(B))]^c$.

This implies that $U \cup [\text{int}(\text{cl}(B))]^c \subseteq \text{int}(\text{cl}(Y)) \subseteq \text{int}(\text{cl}(B))$.

Therefore $U \subseteq \text{int}(\text{cl}(B))$. Since $\text{int}(\text{cl}(B))$ is not contained in$\text{int}(\text{cl}(B))$.

Thus $B$ is gb*-closed set relative to $X$.

Theorem 3-7: Let $A \subseteq Y \subseteq X$ and suppose that $A$ is gb*-closed set in $X$ then $A$ is gb*-closed set relative to $Y$.

Proof: Assume that $A \subseteq Y \subseteq X$ and $A$ is gb*-closed set in $X$. To show that $A$ is gb*-closed set relative to $Y$, let $A \subseteq Y \cap U$ where $U$ is gb-open in $X$. Since $A$ is gb*-closed set in $X$, $A \subseteq U$ implies that $\text{int}(\text{cl}(A)) \subseteq U$, (i.e) $Y \cap \text{int}(\text{cl}(A)) \subseteq Y \cap U$. where $Y \cap \text{int}(\text{cl}(A))$ is interior of closure of $A$ in $Y$. Thus $A$ is gb*-closed set relative to $Y$.

Theorem 3-8: If $A$ is a gb*-closed set and $A \subseteq B \subseteq \text{int}(\text{cl}(A))$ then $B$ is a gb*-closed set.
Proof: Let U be a gb-open set of X, such that B ⊆ U. Then A ⊆ U. Since A is gb*-closed, then int(cl(A)) ⊆ U. Now int(cl(B)) ⊆ int(cl(A)) ⊆ U. Therefore B is gb*-closed set in X.

Theorem 3-9: The intersection of a gb*-closed set and a closed set is a gb*-closed set.

Proof: Let A be a gb*-closed set and F be a closed set. Since A is gb*-closed set, int(cl(A)) ⊆ U whenever A ⊆ U, where U is agb-open set. To show that A ∩ F is gb*-closed set, it is enough to show that int(cl(A ∩ F)) ⊆ U whenever A ∩ F ⊆ U, where U is gb-open set. Let G = X – F then A ⊆ U ∪ G. Since G is open set, U ∪ G is gb-open set and A is gb*-closed set, int(cl(A)) ⊆ U ∪ G. Now int(cl(A ∩ F)) ⊆ int(cl(A)) ∩ int(cl(F)) ⊆ int(cl(A)) ∩ F ⊆ (U ∪ G) ∩ F ⊆ (U ∩ F) ∪ (G ∩ F) ⊆ (U ∩ F) ∪ ∅ ⊆ U. This implies that (A ∩ F) is gb*-closed set.

Theorem 3-10: If A and B are two gb*-closed sets defined for a non-empty set X, then their intersection A ∩ B is gb*-closed set in X.

Proof: Let A and B are two gb*-closed sets in X. Let A ∩ B ⊆ U, U is gp-open set in X. Since A is gb*-closed, int(cl(A)) ⊆ U whenever A ⊆ U, U is g-open set in X. Since B is gb*-closed, int(cl(B)) ⊆ U whenever B ⊆ U, U is g-open set in X. Hence A ∩ B is gb*-closed set.

Remark 3-11: The Union of two gb*-closed sets need not to be gb*-closed set.

Example 3-12: Let X = {a, b, c} with T = {X, ∅, {a}, {c}, {a, c}}. If A = {a}, B = {c} are gb*-closed set in X. then A ∪ B is not a gb*-closed set.

Theorem 3-13: Every gb - closed set is gb* -closed set.

Proof: Assume that A be a gb - closed set in X. and let U be an open set such that A ⊆ U. Since every open set is gb-open set. Then int(cl(A)) ⊆ bcl(A) ⊆ U. hence A is gb*-closed set.

Remark 3-14: The converse of the Theorem [3-13] need not be true as seen by the following example.

Example 3-15: let X = {a, b, c} with T = {X, ∅, {a}}. In this topological space, the subset A = {a, b} is gb* -closed set, but not gb-closed set.

Theorem 3-16: Every gb*-closed set is b-closed set.

Proof: Assume that A is a gb* -closed set in X, and let U be an open set such that A ⊆ U. Since every open set is b-open set and A is gb*-closed set, then int(cl(A)) ⊆ intcl((A)))Ucl(int(A)) ⊆ U. Therefore A is b-closed set in X.
Remark 3-17: The converse of the Theorem [3-16] need not be true as the following example shows.

Example 3-18: let \(X=\{a,b,c\}\) with \(T=\{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}\). In this topological space, the subset \(A=\{a,c\}\) is \(b\)-closed set but not \(gb^*-\) closed set.

Theorem 3-19: Every \(w\)-closed set is \(gb^*\)-closed set.

Proof: Assume that \(A\) is \(w\)-closed set in \(X\), and \(U\) is semi-open set such that \(A \subseteq U\), every semi-open set is \(gb\)-open set then \(cl(A) \subseteq int(cl(A))\) therefore \(A\) is \(gb^*\)-closed set.

Remark 3-20: The converse of the Theorem [3-19] need not be true as seen by the following example.

Example 3-21: let \(X=\{a,b,c\}\) with \(T=\{X, \emptyset, \{a\}, \{c\}, \{a,c\}\}\). In this topological spaces, the subset \(A=\{a\}\) is \(gb^*\)-closed set but not \(w\)-closed set.

Theorem 3-22: Every \(b^*\)-closed set is \(gb^*\)-closed set.

Proof: Assume that \(A\) is a \(b^*-\)closed set in \(X\), and \(U\) is \(b\)-open set such that \(A \subseteq U\), every \(b\)-open set is \(gb\)-open set. Then \(int(cl(A)) \subseteq U\), Therefore \(A\) is \(gb^*\)-closed set.

Remark 3-23: The converse of the Theorem [3-22] need not be true as seen by the following example.

Example 3-24: let \(X=\{a,b,c,d\}\) with \(T=\{X, \emptyset, \{b\}, \{c,d\}, \{b,c,d\}\}\).

In this topological spaces the subset \(A=\{c\}\) is \(gb^*-\)closed set but not \(b^*-\)closed set.

Theorem 3-25: Every \(gb^*\)-closed set is \(g^*b\)-closed set.

Proof: Assume that \(A\) is a \(gb^*-\)closed set in \(X\). Then \(int(cl(A)) \subseteq U\), \(U\) is \(gb\)-open set such that \(A \subseteq U\). Then \(bcl(A) \subseteq intcl(A)\). Since every \(g\)-open set is \(gb\)-open set. Then \(bcl(A) \subseteq U\), \(U\) is \(g\)-open set. Therefore \(A\) is \(g^*b\)-closed set.

Remark 3-26: The converse of the Theorem [3-25] need not be true as seen by the following example.

Example 3-27: let \(X=\{a,b,c\}\) with \(T=\{X, \emptyset\}\). In this topological spaces, the subset \(A=\{a,b\}\) is \(g^*b\)-closed set but not \(gb^*-\)closed set.

Theorem 3-28: Every \(gb^*\)-closed set is \(sg\)-closed set.

Proof: Assume that \(A\) is \(gb^*\)-closed set in \(X\), and \(U\) is \(open\) set such that \(A \subseteq U\) every open set is semi-open set, \(A\) is \(gb^*-\)closed and \(U\) is \(gb\)-closed then \(int(cl(A)) \subseteq A \cup scl(A) \subseteq U\) therefore \(A\) is \(sg\)-closed set.
**Remark 3- 29:** The converse of the Theorem [3-28] need not be true as seen by the following example.

**Example3-30:** let \( X = \{a, b, c\} \), \( T = \{X, \emptyset, \{a, b\}, \{c\}\} \) In this example \( A = \{a, b\} \) is sg-closed set but not gb*-closed set.

**Theorem 3- 31:** Every gb* -closed set is gb-closed set .

**proof:** Assume that \( A \) is gb-closed set in \( X \), and \( U \) is open set such that \( A \subseteq U \), every open set is gb-open set then \( \text{int} (\text{cl} (A)) \subseteq A \cup \beta - \text{closed} \subseteq U \) Therefore \( A \) is gb*-closed set.

**Remark 3- 32:** The converse of the Theorem [3-31] need not be true as seen by the following example.

**Example3-33:** let \( X = \{a, b, c\} \), \( T = \{X, \emptyset, \{b\}, \{b, c\}\} \).

In this example \( A = \{a, b\} \) is gb-closed set but not gb*-closed set.

**Theorem 3- 34:** Every gb*-closed set is b**-closed set .

**proof:** Assume that \( A \) is gb-closed set in \( X \), and \( U \) is open set such that \( A \subseteq U \), every open set is gb-open set then \( \text{int} (\text{cl} (A)) \subseteq \text{cl} (\text{int} (\text{cl} (A))) \cup \text{int} (\text{cl} (\text{int} (A))) \subseteq U \) . Therefore \( A \) is b*-closed set.

**Remark 3- 35:** The converse of the Theorem [3-34] need not be true as seen by the following example.

**Example3-36:** let \( X = \{a, b, c\} \), \( T = \{X, \emptyset, \{a, b\}, \{c\}\} \).

In this topological spaces the subset \( A = \{b, c\} \) is b*-closed set but not gb*-closed set.

---

**diagram (1)**
4. gb*-closed set is independent of other closed sets

In this section, we explain independency of gb*-closed set with some other closed sets.

**Remark 4-1:** The following example shows that the concept of g-closed and gb*-closed sets are independent.

**Example 4-2:** Let \( X = \{a, b, c\} \), \( T = \{X, \emptyset, \{a\}, \{b, c\}\} \). In this topological space, the subset \( A = \{a, b\} \) is g-closed set but not gb*-closed set. And in this topological space, the subset \( B = \{c\} \) is gb*-closed set but not g-closed set.

**Remark 4-3:** The following example shows that the concept of sb*-closed and gb*-closed sets are independent.

**Example 4-4:** Let \( X = \{a, b, c\} \), \( T = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\} \). In this topological space, the subset \( A = \{a, c\} \) is sb*-closed set but not gb*-closed set. And in this topological space, the subset \( B = \{c\} \) is gb*-closed set but not sb*-closed set.

**Remark 4-5:** The following example shows that the concept of g*-closed and gb*-closed sets are independent.

**Example 4-6:** Let \( X = \{a, b, c\} \), \( T = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\} \). In this topological space, the subset \( A = \{a, b\} \) is g*-closed set but not gb*-closed set. And in this topological space, the subset \( B = \{c\} \) is gb*-closed set but not g*-closed set.

**Remark 4-7:** The following example shows that the concept of \( g^\alpha \)-closed and gb*-closed sets are independent.

**Example 4-8:** Let \( X = \{a, b, c\} \) with the topology \( T_1 = \{X, \emptyset, \{a\}, \{c\}\} \). In this topological space, the subset \( A = \{b, c\} \) is \( g^\alpha \)-closed set but not gb*-closed set. And in this topological space, the subset \( B = \{a\} \) is gb*-closed set but not \( g^\alpha \)-closed set.

**Remark 4-9:** The following example shows that the concept of gp-closed and gb*-closed sets are independent.

**Example 4-10:** Let \( X = \{a, b, c\} \) with the topology \( T_1 = \{X, \emptyset, \{a, b\}\} \). In this topological space, the subset \( A = \{a, c\} \) is gp-closed set but not gb*-closed set. For the topology \( T_2 = \{X, \emptyset, \{a\}, \{b\}\} \) topological, the subset \( B = \{b\} \) is gb*-closed set but not gp-closed set.

**Remark 4-11:** The following example shows that the concept of wg-closed and gb*-closed sets are independent.

**Example 4-12:** Let \( X = \{a, b, c\} \) with the topology \( T_1 = \{X, \emptyset, \{a\}\} \). In this topological space, the subset \( A = \{a, b\} \) is wg-closed set but not gb*-closed set. For the topology \( T_2 = \{X, \emptyset, \{a\}, \{c\}\} \) topological, the subset \( B = \{a\} \) is gb*-closed set but not wg-closed set.
Reference


حول المجموعات المغلقة بالنمط –* 

زيتة طه الحويز
قسم الرياضيات/ كلية التربية للبنات/ جامعة تكريت

استلم البحث في: 22/كانون الأول/2014، قبل البحث في: 20/أيلول/2015

الخلاصة

يعرض هذا البحث دراسة مفهوم جديد من المجموعات المغلقة يسمى المجموعات الممثالة المغلقة *بُب، في الفضاءات التوبولوجية كما نقوم بدراسة بعض الخصائص الأساسية، ودراسة العلاقات بينها وبين المجموعات المغلقة في الفضاء التوبولوجي.

الكلمات المفتاحية: المجموعات الممثالة المغلقة -*بُب، والمجموعات الممثالة المغلقة -ب، والمجموعات الممثالة المغلقة.