On 2-Absorbing Submodules

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Abstract
Let R be a commutative ring with \( 1 \neq 0 \) and M is a unitary R-module. In this paper, our aim is to continue studying 2-absorbing submodules which are introduced by A.Y. Darani and F. Soheilina. Many new properties and characterizations are given.

Key words: prime submodule, 2-absorbing submodule, quasi-prime submodule, multiplication module, P-primary submodule, pure submodule.

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Introduction

The concept of 2-absorbing ideal which is a generalization of prime ideal was introduced by Ayman Badawi, where "a nonzero proper ideal I of R is called a 2-absorbing ideal of R if whenever a, b, c ∈ R, abc ∈ I then ab ∈ I or ac ∈ I or bc ∈ I" [1]. This definition can obviously be made for any proper ideal. A. Y. Darani and F. Soheilnia in [2] introduced the concept of 2-absorbing submodule where "a proper submodule N of M is called 2-absorbing submodule of M if whenever a, b ∈ R, m∈ M and abm ∈ N, then am ∈ N or bm ∈ N or ab ∈ (N:M)"

Our concern in this paper is to give a comprehensive study of 2-absorbing submodule, where we give many new properties and characterizations.

1. 2-Absorbing Submodules – Basic Properties

"Definition 1.1:

Let M be an R-module. N a proper submodule of M is called 2-absorbing submodule if whenever a,b ∈ R , m ∈ M and abm ∈ N, then ab ∈ (N:M) or am ∈ N or bm ∈ N."[2]

Note that an 2-absorbing ideal of a ring R is a 2-absorbing submodule of the R-module R.

Remarks and Examples 1.2:

(1) "The intersection of each pair of distinct prime submodules of R-module M is 2-absorbing" [2]

(2) It is clear that every prime submodule is 2-absorbing. However the converse is not true in general, for example:

Consider Z_6 as Z-module, (0) not prime submodule of Z_6 since 2.3 = 0 ∈ (0) but 3 ∉ (0) and 2 ∉ ((0) ∩ Z_6) but (0) ∩ (3) is 2-absorbing submodule of Z_6 as Z-module by part (1)

(3) It is clear that every quasi-prime submodule is 2-absorbing, where" a proper submodule N of M is called quasi-prime submodule if whenever a, b ∈ R, m∈ M and abm ∈ N, then a m ∈ N or b m ∈ N"[3]

However a 2-absorbing submodule may not be quasi-prime, as the following example shown:

Consider the Z-module Z. The submodule N =4Z is a 2-absorbing submodule of Z since, if a,b,c ∈ Z with abc ∈ 4Z=N, then at least two of a,b,c are even. Hence either ab ∈ N or ac ∈ N or bc ∈ N

But 4Z is not quasi-prime, since 2.2.1 ∈4Z but 2.1 ∉4Z.

(4) Let N,W be two submodules of an R-module M and W < N .If N is 2-absorbing submodule of M then it is not necessary that W is 2-absorbing submodule of M, for example in Z_{24} as Z-module. Take N=(6), W=(12). Since N=(2) ∩ (3) and both of them are prime submodules then N is 2-absorbing submodule by part (1). But 2.2.3 ∈ W, 2.3 ∉ W and 2.2.4 ∈ (W ∩ Z_{24}) =12Z. Thus W is not 2-absorbing submodule in M.
5. Let \( N, W \) be two submodules of an \( R \)-module \( M \) and \( N < W \). If \( N \) is 2-absorbing submodule of \( M \), then \( N \) is a 2-absorbing submodule of \( W \)

**Proof:**

If \( W = M \), then nothing to prove.

Let \( a b x \in N \), where \( a, b \in R \), \( x \in W \). Since \( x \in W \) then \( x \in M \). But \( N \) is 2-absorbing submodule of \( M \), so either: \( a x \in N \) or \( b x \in N \) or \( a b \in (N:M) \) and since \( N < W \) implies \( (N:M) \leq (N:W) \), then either \( a x \in W \) or \( b x \in W \) or \( a b \in (N:W) \).

Hence \( N \) is 2-absorbing in \( W \).

\( \square \)

6. The sum of 2-absorbing submodules is not necessarily 2-absorbing, for example:

Let \( N_1 = 2Z \), \( N_2 = 3Z \), each of \( N_1 \) and \( N_2 \) is 2-absorbing submodule in the \( Z \)-module \( Z \), but \( N_1 + N_2 = Z \) which is not 2-absorbing.

7. Let \( N \) and \( W \) be two submodules of an \( R \)-module \( M \) such that \( N \cong W \)

If \( N \) is 2-absorbing submodule, it is not necessary that \( W \) is 2-absorbing submodule as the following example explains this:

Consider the \( Z \)-module \( Z \), the submodule \( 2Z \) is 2-absorbing submodule but \( 2Z \cong 30Z \) and \( 30Z \) is not 2-absorbing since \( 2.3.5 = 30 \in 30 \) but \( 2.5 \notin 30Z \) and \( 3.5 \notin 30Z \) and \( 2.3 = 6 \notin 30Z \).

8. The intersection of two 2-absorbing submodule need not be 2-absorbing submodule for example: \( 6Z \) and \( 5Z \) are 2-absorbing submodule in the \( Z \)-module \( Z \), but \( 6Z \cap 5Z = 30Z \) which is not 2-absorbing.

9. Let \( N \) be a 2-absorbing submodule of \( M \), then for each \( A \subseteq M \), either \( A \subseteq N \) or \( A \cap N \) is a 2-absorbing submodule of \( A \)

**Proof:**

Suppose \( A \nsubseteq N \). Then \( A \cap N \nsubseteq A \).

Let \( abx \in A \cap N \), and \( x \in A \), \( a,b \in R \). So \( abx \in N \). Since \( N \) is 2-absorbing, either \( ax \in N \) or \( bx \in N \) or \( ab \in (N:M) \), then \( ax \in A \cap N \) or \( bx \in A \cap N \) or \( ab \in (A \cap N:A) \).

**Proposition 1.3:**

Let \( \varphi : M \to M' \) be an \( R \)-epimorphism. If \( W \) is 2-absorbing submodule of \( M' \), then \( \varphi^{-1}(W) \) is 2-absorbing submodule of \( M \).

**Proof:**

It is straight forward so it is omitted

**Proposition 1.4:**

Let \( f : M \to M' \) be an epimorphism, \( N < M \) such that \( \ker f \subseteq N \), then \( N \) is 2-absorbing submodule of \( M \) if and only if \( f(N) \) is 2-absorbing submodule of \( M' \).
Proof:

\((\Rightarrow)\) Let \(ab \in \mathcal{f}(N)\), where \(\mathfrak{m} \in \mathcal{M}, a,b \in \mathbb{R}\). \(\mathfrak{m} = f(m)\) for some \(m \in \mathcal{M}\), since \(f\) is onto.

Then \(abf(m) \in \mathcal{f}(N)\), so \(abf(m) = f(n)\) for some \(n \in \mathcal{N}\) and hence \(f(abm) = f(n) = 0\). Thus we get that \(abm - n \in \ker f \subseteq \mathcal{N}\) which implies that \(abm \in \mathcal{N}\). But \(\mathcal{N}\) is 2-absorbing so either \(am \in \mathcal{N}\) or \(bm \in \mathcal{N}\) or \(ab \in (\mathcal{N} \cap \mathcal{M})\).

If \(am \in \mathcal{N}\), then \(f(am) \in \mathcal{f}(\mathcal{N})\), that is \(a \in \mathcal{f}(\mathcal{M})\) so \(a \mathfrak{m} \in \mathcal{f}(N)\).

Similarly, \(bm \in \mathcal{N}\) implies that \(bm \in \mathcal{f}(N)\).

If \(ab \in (\mathcal{N} \cap \mathcal{M})\), then \(abM \subseteq \mathcal{N}\) and so \(f(abM) \subseteq \mathcal{f}(\mathcal{N})\) which implies that \(abM \subseteq \mathcal{f}(\mathcal{N})\) and we get that \(ab \in (f(\mathcal{N}) \cap \mathcal{M})\).

\((\Leftarrow)\) Let \(ab \in \mathcal{N}\) then \(f(abm) \in \mathcal{f}(\mathcal{N})\) so \(abf(m) \in \mathcal{f}(\mathcal{N})\). Since \(f(\mathcal{N})\) is 2-absorbing either \(af(m) \in f(\mathcal{N})\) or \(bf(m) \in f(\mathcal{N})\) or \(ab \in (f(\mathcal{N}) \cap \mathcal{M})\).

1) If \(af(m) \in f(\mathcal{N})\) then \(f(am) = f(n)\) for some \(n \in f(\mathcal{N})\), hence \(am-n \in \ker f \subseteq \mathcal{N}\), so \(am \in \mathcal{N}\).

2) If \(bf(m) \in f(\mathcal{N})\) then similarly that \(bm \in \mathcal{N}\).

3) If \(ab \in (f(\mathcal{N}) \cap \mathcal{M})\) then \(abM \subseteq \mathcal{N}\) so \(abf(x) \in \mathcal{f}(\mathcal{N})\) for each \(x \in \mathcal{M}\) so that \(f(abx) = f(n)\) for some \(n \in \mathcal{N}\) and hence \(abx \in \mathcal{N}\) for each \(x \in \mathcal{M}\). Thus \(ab \in (\mathcal{N} \cap \mathcal{M})\).

By using Proposition 1.4 we can get the following result which is given in [2] as a direct consequence.

"Corollary 1.5:

Let \(R\) be a ring, \(M\) an \(R\)-module and \(\mathcal{N}, \mathcal{K}\) submodules of \(M\) with \(\mathcal{K} \subseteq \mathcal{N}\). Then \(\mathcal{N}\) is a 2-absorbing submodule of \(M\) if and only if \((\mathcal{N} : \mathcal{R} \mathcal{M})\) is a 2-absorbing submodule of \((\mathcal{M} : \mathcal{R})\).

A. Y. Darani and F. Soheilina in [2] introduced the following:

"Proposition 1.6:

Let \(R\) be a commutative ring, \(M\) is a cyclic \(R\)-module and \(\mathcal{N}\) is a submodule of \(M\). Then \(\mathcal{N}\) is a 2-absorbing submodule of \(M\) if and only if \((\mathcal{N} : \mathcal{R} \mathcal{M})\) is a 2-absorbing ideal of \(R\).

Sh. Payrovi and S. Babaei in [4] and S. Moradi and A. Azizi in [5] introduced the following:

"Theorem 1.7:

"If \(\mathcal{N}\) is a 2-absorbing submodule of \(M\), then \((\mathcal{N} : \mathcal{R} \mathcal{M})\) is a 2-absorbing ideal of \(R\)."

Sh. Payrovi, Babaei in [4] proved the following: "Let \(R\) be a Noetherian ring, \(M\) a finitely generated multiplication \(R\)-module and \(\mathcal{N}\) a proper submodule of \(M\) such that \(\text{Ass}_R(M/\mathcal{N})\) is a totally ordered set. If \((\mathcal{N} : \mathcal{R} \mathcal{M})\) is a 2-absorbing ideal of \(R\), then \(\mathcal{N}\) is a 2-absorbing submodule of \(M\)."
However we get the same conclusion under the class of multiplication modules. Before giving our result, recall that "An $R$-module is called a multiplication module if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $IM = N$. Equivalently, $M$ is a multiplication module if for every submodule $N$ of $M$, $N = (N :_RM)M$. [6]

**Theorem 1.8:**

Let $M$ be a multiplication $R$-module and $N$ is a proper submodule of $M$, If $(N :_RM)$ is a 2-absorbing ideal of $R$, then $N$ is 2-absorbing submodule of $M$

**Proof:**

Let $abm \in N$ where $a,b \in R$, $m \in M$ then $ab(m) \subseteq N$.

But $(m) = IM$ for some $I \subseteq R$ since $M$ is a multiplication $R$-module, so $abIM \subseteq N$.

Hence $abI \subseteq (N :_RM)$, so we get that $(a)(b)I \subseteq (N :_RM)$. Since $(N :_RM)$ is a 2-absorbing ideal, therefore $(a)I \subseteq (N :_RM)$ or $(b)I \subseteq (N :_RM)$ or $(a)(b) \subseteq (N :_RM)$ by [1]

1) If $(a)I \subseteq (N :_RM)$, then $(a)IM \subseteq N$ so $(a)(m) \subseteq N$ thus $am \in N$

2) If $(b)I \subseteq (N :_RM)$, then similarly $bm \in N$.

3) If $(a)(b) \subseteq (N :_RM)$, then $ab \in (N :_RM)$

**Corollary 1.9:**

Let $M$ be a multiplication $R$-module and $N$ is a proper submodule of $M$, then $N$ is 2-absorbing submodule of $M$ if and only if $(N :_RM)$ is 2-absorbing ideal.

**Remark 1.10:**

The condition $M$ is a multiplication $R$-module can't be dropped from Theorem 1.8.

Consider the following example:

Let $M$ be the $Z$-module $Z_{p^{\infty}}$ and $N = (0)$. $N$ is not 2-absorbing since:

$p.p. \left( \frac{1}{p^2} + Z \right) = 0$ but $p. \left( \frac{1}{p^2} + Z \right) \neq 0$ and $P^2 \notin (\frac{1}{p^2} + Z :_R Z) = (0)$, Also notice that $(N : Z_{p^{\infty}}) = (0)$, and $(0)$ is a prime ideal in $Z$, so $(0)$ is 2-absorbing ideal in $Z$.

Recall that "A proper submodule $N$ of an $R$-module $M$ is called a $P$-primary submodule of $M$ if whenever $a \in R$ and $m \in M$ and $am \in N$, then $m \in N$ or $a \in \sqrt{N : M}$ = $P$ " [7]

Darani in [7] proved the following:

"Let $N$ be a $P$-primary submodule of a cyclic $R$-module. Then $N$ is 2-absorbing if and only if $(pM)^2 \subseteq N$"

However we improve this Theorem as follows.

**Proposition 1.11:**

Let $N$ be a $p$-primary submodule of a multiplication $R$-module $M$. Then $N$ is 2-absorbing if and only if $(pM)^2 \subseteq N$ (where $(pM)^2 = p^2M$).
Proof:

\( (\Rightarrow) \) Since \( M \) is a multiplication \( R \)-module, \( M \)-rad\( N = \sqrt{N:M} \) \( M \) by [7, Th 2.10]

But \( N \) is \( P \)-primary, so \( P = \sqrt{N:M} \) is a prime ideal. Thus \( M \)-rad\( N = PM \).

It follows that \( P^2 \subseteq (N:M) \) by [1] Th. 2.4. Thus \( P^2M \subseteq N \).

Hence \( (PM)^2 \subseteq N \)

\( (\Leftarrow) \) Let \( abm \in N \) where \( a, b \in R, m \in M \)

Assume that \( am \in N \) and \( bm \notin N \). As \( N \) is primary with \( abm \in N \) and \( bm \in N \), we get \( a \in \sqrt{N:M} = P \)

Also \( abm \in N \) and \( am \in N \), we get \( b \in \sqrt{N:M} = P \)

Thus \( ab \in (N:M) \) and consequently \( N \) is 2-absorbing submodule of \( M \).

Recall that "A proper submodule \( N \) of an \( R \)-module \( M \) is called a 2-absorbing primary submodule of \( M \) if whenever \( a; b \in R \) and \( m \in M \) and \( abm \in N \), then \( am \in M \)-rad\( (N) \) or \( bm \in M \)-rad\( (N) \) or \( ab \in (N :_R M)" [8]

It is clear that 2-absorbing submodule implies 2-absorbing primary, but the converse may be not hold, as the example shows.

Let \( M \) be the \( Z \)-module \( Z \), let \( N=8Z \), so \( N \) is 2-absorbing primary, but it is not 2-absorbing submodule.

Babaei in [4] proved the following:

"Let \( R \) be a Noetherian ring, \( I \) a 2-absorbing ideal of \( R \), and \( M \) a faithful multiplication \( R \)-module such that Ass\( R(M /IM) \) is a non-empty totally ordered set. Then \( abm \in IM \) implies that \( am \in IM \) or \( bm \in IM \) or \( ab \in I \) whenever \( a; b \in R \) and \( m \in M \)" [4], that \( IM \) is 2-absorbing. Where \( R \) is a commutative ring with nonzero identity.

However we get the same conclusion of this theorem but with less conditions and also we give a simple proof.

Proposition 1.12:

Suppose \( M \) is a finitely generated multiplication \( R \)-module. If \( I \) is 2-absorbing ideal of \( R \) such that \( \text{ann} M \subseteq I \), then \( IM \) is 2-absorbing submodule of \( M \).

Proof:

Let \( abm \in IM \) where \( a, b \in R, m \in M \), hence \( ab(m) \in IM \). Since \( M \) is multiplication, then \( (m)=JM \) for some ideal \( J \) of \( R \). Thus \( abJM \subseteq IM \) and so \( ab \subseteq I+\text{ann} M = I \). But \( I \) is a 2-absorbing ideal of \( R \), so either \( ab \in I \) or \( aJ \subseteq I \) or \( bJ \subseteq I \), it follows that \( ab \in (IM \cap M) \) or \( aJM \subseteq IM \) or \( bJM \subseteq IM \); that is either \( ab \in (IM \cap R) \) or \( a(m) \subseteq IM \) or \( b(m) \subseteq IM \) Thus \( ab \in (IM \cap M) \) or \( am \in IM \) or \( bm \in IM \) and so \( IM \) is a 2-absorbing submodule of \( M \).
Corollary 1.13:

Suppose $M$ is a faithful finitely generated multiplication $R$-module. If $I$ is a 2-absorbing ideal of $R$, then $IM$ is a 2-absorbing submodule of $M$.

Proof:

It follows directly by Proposition 1.12.

Corollary 1.14:

Let $M$ be a faithful finitely generated multiplication $R$-module. Then every proper submodule of $M$ is 2-absorbing if and only if every proper ideal of $R$ is 2-absorbing.

Proof:

$(\Leftarrow)$ It follows directly by Corollary (1.13).

$(\Rightarrow)$ Let $I$ be a proper ideal of $R$. Then $N=IM$ is a proper submodule of $M$.

So it is 2-absorbing and hence by Theorem (1.7), $(N:M)$ is 2-absorbing ideal. But $M$ is faithful finitely generated multiplication $R$-module, so $(N:M)=I$ by ([6],Th.3.1).

"Proposition 1.15: [4,Th 2.4]"

Let $M$ be an $R$-module, $N$ a proper submodule of $M$, if $N$ is 2-absorbing then $(N_R(m))$ is 2-absorbing ideal for each $m \in M-N$.

Proof:

First $(N_R(m)) \neq R$ for any $m \notin N$.

Let $abc \in (N_R(m))$ then $ab(cm) \in N$, but $N$ is 2-absorbing submodule then

$h(a, b, c) \in (N_R(m))$ so that $acm \in N$ or $bcm \in N$ or $abM \subseteq N$ that is $ac \in (N_R(m))$ or $bc \in (N_R(m))$ or $ab \in (N_R(m))$ hence $(N_R(m))$ is 2-absorbing ideal.

Recall that "a submodule $N$ of $M$ is called a pure submodule of an $R$-module $M$ if $IM \cap N = IN$ for any ideal $I$ of $R$" [9]

Proposition 1.16:

Let $N$ be a proper pure submodule of an $R$-module $M$, If $(0)$ is a 2-absorbing submodule of $M$, then $N$ is 2-absorbing.

Proof:

Let $abm \in N$ where $a, b, m \in M$.

Put $I = (ab)$ then $abm \in IM \cap N$, but $IM \cap N = IN$, so $abm = abn$ for some $n \in N$, then $ab(m - n) = 0$, but $(0)$ is 2-absorbing then $a(m - n) = 0$ or $b(m - n) = 0$ or $ab \in annM \subseteq (N : M)$. So we get $am = an \in N$ or $bm = bn \in N$ or $ab \in (N : M)$.

Thus $N$ is a 2-absorbing submodule.

2. 2-absorbing Submodules Characterizations.

In this section we give some characterizations of 2-absorbing submodules. We start with the following proposition:
Proposition 2.1:
Let $N$ a proper submodule of an $R$-module $M$. Then $N$ is a 2-absorbing submodule of $M$ if and only if $abK \subseteq N$ for some $a, b \in R, K \subseteq M$ implies $ab \in (N:M)$, or $aK \subseteq N$ or $bK \subseteq N$.

Proof:
Suppose that $ab \notin (N:M)$ and $aK \not\subseteq N$ and $bK \not\subseteq N$. Then there exist $m_1, m_2$ in $K$ such that $am_1 \notin N$ and $bm_1 \notin N$.

Since $am_1 \in N$ and $ab \notin (N:M)$, we get $bm_1 \in N$.

Also since $abm_2 \in N$ and $ab \notin (N:M)$, we get $am_2 \in N$.

Now, since $ab(m_1 + m_2) \in N$ and $ab \notin (N:M)$ we have $a(m_1 + m_2) \in N$ or $b(m_1 + m_2) \in N$.

If $a(m_1 + m_2) \in N$; i.e. $am_1 + am_2 \in N$ and since $am_2 \in N$ we get $am_1 \in N$ which is contradiction!

If $b(m_1 + m_2) \in N$; i.e. $bm_1 + bm_2 \in N$ and since $bm_2 \in N$ we get $bm_1 \in N$ which is contradiction!

Then either $ab \in (N:M)$ or $aK \subseteq N$ and $bK \subseteq N$.

The converse is clear.

The following theorem give a useful characterization of 2-absorbing submodule.

Theorem 2.2:
Let $N$ a proper submodule of an $R$-module $M$, then the following statement are equivalent:

1. $N$ is a 2-absorbing submodule of $M$
2. If $I J K \subseteq N$, for some ideal $I$ and $J$ of $R$ and some submodule $K$ of $M$ then either $I K \subseteq N$ or $J K \subseteq N$ or $I J \subseteq (N:M)$.

Proof: $(1) \Rightarrow (2)$
Suppose $N$ is a 2-absorbing submodule of $M$ and $I J K \subseteq N$ for some ideals $I$ and $J$ of $R$ and some submodule $K$ of $M$ and $I J \not\subseteq (N:M)$.

To show that $I K \subseteq N$ or $J K \subseteq N$.

Suppose $I K \not\subseteq N$ and $J K \not\subseteq N$. then there exist $a_1 \in I$ and $a_2 \in J$ such that $a_1 K \not\subseteq N$ and $a_2 K \not\subseteq N$.

But $a_1 a_2 K \not\subseteq N$ and neither $a_1 K \not\subseteq N$ nor $a_2 K \not\subseteq N$ and $N$ is 2-absorbing, so we have $a_1 a_2 \in (N:M)$ by Proposition (2.1).

Since $I J \not\subseteq (N:M)$, then there exist $b_1 \in I$ and $b_2 \in J$ such that $b_1 b_2 \notin (N:M)$.

But $b_1 b_2 K \not\subseteq N$, so we have $b_1 K \not\subseteq N$ or $b_2 K \not\subseteq N$ by Proposition (2.1).

Now we have the following cases:

Case (1) $b_1 K \subseteq N$ and $b_2 K \subseteq N$
Since \(a_1b_2 \subseteq N\) and \(b_2 \not\subseteq N\) and \(a_1 \not\subseteq N\) so that \(a_1b_2 \in (N:M)\) by Proposition (2.1).

Since \(b_1K \subseteq N\) and \(a_1K \not\subseteq N\), we conclude \((a_1+b_1)K \not\subseteq N\). On the other hand, \((a_1+b_1)b_2 K \subseteq N\) and neither \((a_1+b_1)K \subseteq N\) nor \(b_2 K \subseteq N\), we get that \((a_1+b_1)b_2 \in (N:M)\) by Proposition (2.1).

But \((a_1+b_1)b_2 = a_1b_2 + b_1b_2 \in (N:M)\) and \(a_1b_2 \in (N:M)\), we get \(b_1b_2 \in (N:M)\) which is a contradiction!

**Case(2)** If \(b_2K \subseteq N\) and \(b_1K \not\subseteq N\). By a similar argument of case (1), we reach to a contradiction!

**Case(3)** \(b_1K \subseteq N\) and \(b_2K \subseteq N\)

Since \(b_2K \subseteq N\) and \(a_2K \not\subseteq N\), we conclude \((a_2+b_2)K \not\subseteq N\). But \(a_1(a_2+b_2)K \subseteq N\) and neither \(a_1K \subseteq N\) nor \((a_2+b_2)K \subseteq N\), hence \(a_1(a_2+b_2) \in (N:M)\) by Proposition (2.1).

Since \(a_1a_2 \in (N:M)\) and \(a_2 + a_1b_2 \in (N:M)\), we have \(a_1b_2 \in (N:M)\). Since \((a_1+b_1)a_2 \subseteq N\) and neither \(a_2 \subseteq N\) nor \((a_1+b_1)K \subseteq N\), we conclude \((a_1+b_1)a_2 \in (N:M)\) by Proposition (2.1).

But \((a_1+b_1)a_2 = a_1a_2 + a_1b_2 \in (N:M)\) and since \(a_1a_2 \in (N:M)\), we get \(a_1b_2 \in (N:M)\). Now, since \((a_1+b_1)(a_2+b_2)K \subseteq N\) and neither \((a_1+b_1)K \subseteq N\) nor \((a_2+b_2)K \subseteq N\), we have \((a_1+b_1)(a_2+b_2) = a_1a_2 + a_2b_1 + a_1b_2 + b_1b_2 \in (N:M)\) by Proposition (2.1).

But \(a_1a_2, a_1b_2, b_1a_2 \in (N:M)\), so \(b_1b_2 \in (N:M)\) which is a contradiction! Consequently \(I_1K \subseteq N\) or \(I_2K \subseteq N\).

(2) \implies (1) It is clear.

Recall that \(\). for any two submodules \(N\), \(K\) of a multiplication \(R\)-module \(M\) with \(N = I_1M\) and \(K = I_2M\) for some ideals \(I_1\) and \(I_2\) of \(R\). The product \(N\) and \(K\) denoted by \(NK\) is defined by \(NK = I_1I_2M\).[2]

By using this definition of product of submodules , we give the following characterization of 2-absorbing submodules .

**Theorem 2.3:**

Let \(N\) be a proper submodule of a multiplication \(R\)-module \(M\), then

\(N\) is a 2-absorbing submodule of \(M\) if and only if \(N_1N_2N_3 \subseteq N\) implies that \(N_1N_2 \subseteq N\) or \(N_1N_3 \subseteq N\) or \(N_2N_3 \subseteq N\), where \(N_1\), \(N_2\), \(N_3\) are submodules of \(M\).

**Proof:**

\((\Rightarrow)\) Since \(M\) is multiplication , then \(N_1= I_1M\) , \(N_2= I_2M\) , \(N_3= I_3M\) for some ideals \(I_1\) , \(I_2\) , \(I_3\) of \(R\) . It follows that :

\(N_1N_2N_3 = I_1I_2I_3M \subseteq N\) . Hence \(I_1I_2I_3 \subseteq (N:M)\) . But \(N\) is 2-absorbing submodule of \(M\) implies that \((N:M)\) is 2-absorbing ideal by [Theorem 1.7].

So by [1 ,Th 2.13] either \(I_1I_2 \subseteq (N:M)\) or \(I_1I_3 \subseteq (N:M)\) or \(I_2I_3 \subseteq (N:M)\) . Then \(I_1I_2M \subseteq N\) or \(I_1I_3M \subseteq N\) or \(I_2I_3M \subseteq N\) . Thus \(N_1N_2 \subseteq N\) or \(N_1N_3 \subseteq N\) or \(N_2N_3 \subseteq N\).
Let $I_1, I_2, K \subseteq \mathbb{N}$ where $I_1, I_2$ are ideals of $R$ and $K \subseteq M$. Since $M$ is a multiplication module, $K = JM$ for some $J$ of $R$. Thus $I_1, I_2, J \subseteq \mathbb{N}$. Put $N_1 = I_1M$, $N_2 = I_2M$.

It follows that $N_1, N_2 \subseteq \mathbb{N}$. So by hypotheses either $N_1 \subseteq N$ or $N_2 \subseteq N$, or $N_1, N_2 \subseteq N$. Then $1:J \subseteq \mathbb{N}$ or $I_1, I_2 \subseteq (N:M)$; that is $I_1 \subseteq N$ or $I_2 \subseteq N$ or $I_1, I_2 \subseteq (N:M)$

Thus $N$ is a 2-absorbing submodule of $M$.

**Proposition 2.4:**

Let $N$ be a proper submodule of an $R$-module $M$. The following statements are equivalent:

1. $N$ is a 2-absorbing submodule of $M$
2. $(N: I_M I)$ is 2-absorbing, for each ideal $I$ of $R$ with $IM \not\subseteq N$
3. $(N: I_M (r))$ is 2-absorbing submodule for each $r \in R$ with $rM \not\subseteq N$

**Proof:**

(1) $\Rightarrow$ (2) Let $I$ be an ideal of $R$ with $IM \not\subseteq M$, then $(N: I_M I)$ is a proper submodule of $M$.

Let $a, b, m \in (N: I_M I)$, where $a, b \in R$, $m \in M$. Then $ab(IM) \subseteq N$.

But $N$ is a 2-absorbing submodule of $M$, so by Proposition (2.1), either $a(IM) \subseteq N$ or $b(IM) \subseteq N$ or $ab \in (N: I_M I)$.

Hence either $am \in (N: I_M I)$ or $bm \in (N: I_M I)$ or $ab \in (N: I_M I)$.

Thus $(N: I_M I)$ is a 2-absorbing submodule.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Take $r = 1$ then $(N: I_M (1)) = N$, so $N$ is 2-absorbing.

Now we have the following:

**Theorem 2.5:**

Let $N$ be a proper submodule of an $R$-module $M$. Consider the following statements:

1. $N$ is a 2-absorbing submodule of $M$
2. For each $a, b \in R$, $m \in M$. If $abm \not\in N$, then $(N: abm) = (N: am) \cup (N: bm)$
3. For each $a, b \in R$, $m \in M$. If $abm \not\in N$, then $(N: abm) = (N: am)$ or $(N: abm) = (N: bm)$

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and if $M$ is cyclic, then (3) $\Rightarrow$ (1).

**Proof:**

(1) $\Rightarrow$ (2)

Let $c \in (N: abm)$ then $abcm \in N$. Since $abm \not\in N$ and $N$ is 2-absorbing, so by (1.2), either $acm \in N$ or $bcm \in N$. Then $c \in (N: am)$ or $c \in (N: bm)$.

Thus $(N: abm) \subseteq (N: am) \cup (N: bm)$. To prove reverse inclusion:

Let $c \in (N: am) \cup (N: bm)$, then $c \in (N: am)$ or $c \in (N: bm)$ so that $acm \in N$ or $bcm \in N$.

It follows that $abcm \in bN \subseteq N$ or $abcm \in aN \subseteq N$.

Thus $abcm \in N$ and hence $c \in (N: abm)$.
Then \((N:am) \cup (N:bm) \subseteq (N:abm)\). So we get \((N:abm) = (N:am) \cup (N:bm)\).

(2) \(\Rightarrow\) (3) Since \((N:abm)\) is an ideal of \(R\), and \((N:abm) = (N:am) \cup (N:bm)\).

So either \((N:bm) \subseteq (N:am)\) or \((N:am) \subseteq (N:bm)\). Then \((N:abm) = (N:am)\) or \((N:abm) = (N:bm)\).

(3) \(\Rightarrow\) (1) Now suppose that \(M = (m_1)\) for some \(m_1 \in M\).

Let \(abm \in N\). But \(m = rm_1\) for some \(r \in R\) then \(abrm_1 \in N\) and suppose that \(a b \in (N:M) = (N: (m_1))\), \(abm \notin N\) and \(r \in (N:abm)\).

By (3) \((N:abm) = (N:a m_1)\) or \((N:ab m_1) = (N:b m_1)\). Since \(r \in (N:abm)\), so either \(r \in (N:a m_1)\) or \(r \in (N:b m_1)\). Then \(a rm_1 \in N\) or \(b rm_1 \in N\), we get \(a m \in N\) or \(b m \in N\). Hence \(N\) is 2-absorbing \(\blacksquare\)

**References**

الخلاصة

لتكن R حلقة ابدالية ذات حلاية 1 ≠ 0 و M مقاس احادي على الحلقة R. هدفنا في هذا البحث الاستمرار في دراسة المقاسات الجزئية من النمط المستوحده على 2 والتي قدمت قبل الباحثين دارياني وسيريني. العديد من الخواص والتمييزات لهذا المفهوم قد اعطت

الكلمات المفتاحية : المقاسات الجزئية الأولية ، المقاسات الجزئية المستوحده على2 ، المقاسات الجزئية شبه الأولية ، المقاسات الجدائية ، المقاسات الجزئية الابتدائية ، المقاسات الجزئية النقية.