2-Regular Modules II

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Abstract

An R-module M is called a 2-regular module if every submodule N of M is 2-pure submodule, where a submodule N of M is 2-pure in M if for every ideal I of R, I^2M \cap N = I^2N, [1].

This paper is a continuation of [1]. We give some conditions to characterize this class of modules, also many relationships with other related concepts are introduced.

Key Words: 2-pure submodules, 2-regular modules, pure submodule, regular modules.
0- Introduction
Throughout this paper, R is a commutative ring with identity and all R-modules are unitary. A submodule N of an R-module M is called 2-pure submodule if for every ideal I of R, $I^2M \cap N = I^2N$. If every submodule of M is 2-pure, then M is said to be 2-regular module. This work consists of two sections. In the first section we give some properties of 2-regular rings. Next we present a characterization of 2-regular modules. In the second section we illustrate some relationships between the concept 2-regular modules and other modules such as semiprime divisible, projective and multiplication modules.

1- 2-Regular Modules
In this section, we first define 2-regular rings and study some of its properties. Next we consider some conditions to characterize 2-regular modules.

Definition (1.1): [1]
An ideal I of a ring R is called 2-pure ideal of R if for each ideal J of R, $J^2 \cap I = J^2I$.
If every ideal of a ring R is 2-pure ideal, then we say R is 2-regular ring.

Remarks and Examples (1.2):
(1) It is clear every (von Neumann) regular ring is 2-regular ring, but the converse is not true, for example: the ring $Z_4$ is 2-regular ring, since every ideal of $Z_4$ is 2-pure. But $Z_4$ is not regular since the ideal $\{0,2\}$ is not pure because $\{0,2\} \cap \{0,2\} = \{0,2\}$, on the other hand $\{0,2\} \cdot \{0,2\} = \{0\}$ implies $\{0,2\} \cap \{0,2\} \neq \{0,2\} \cdot \{0,2\}$.
(2) It is clear that $\{0\}$ and R are always 2-pure ideals of any ring R.
(3) Every field is 2-regular ring.
(4) Let R be an integral domain. If R is 2-regular ring, then R is a field.
Proof:
Let I be an ideal of R. Since R is 2-regular ring then $J^2 \cap I = J^2I$ for every ideal J of R. If we take $J = I$ implies $I^2 = I^3$. Thus for each element $0 \neq a \in R$, $<a>^2 = <a>^3$, hence $a^2 < a >^3$. Let $a^2 = r a^3$ for some $r \in R$, then $a^2(1 - ra) = 0$ but R is domain and $a \neq 0$ implies $1 - ra = 0$, thus $1 = ra$. Therefore $a$ is an invertible element of R. Thus R is a field.
(5) If R is a 2-regular ring then every prime ideal of R is a maximal ideal.
Proof:
Let I be a prime ideal of R. Since R is a 2-regular ring then $\frac{R}{I}$ is 2-regular by [1,Cor.3.2]. But $\frac{R}{I}$ is a domain since I is a prime ideal. Thus $\frac{R}{I}$ is a field by the above remark. Therefore I is a maximal ideal.

(6) Every 2-regular ring is nearly regular, where a ring R is called nearly regular if $\frac{R}{J(R)}$ is regular ring, see [2], where $J(R)$ = the intersection of all maximal ideals of R.
Proof:
Let $R$ be a 2-regular ring. Then $\frac{R}{J(R)}$ is 2-regular by corollary (1.2.3). So by above remark (5), every prime ideal of $\frac{R}{J(R)}$ is a maximal ideal and since $J\left(\frac{R}{J(R)}\right) = 0$, therefore by [3], $\frac{R}{J(R)}$ is regular.

**Proposition (1.3):**

Let $M$ be 2-regular $R$-module then for every element $x$ of $M$ and every element $r \in R$, $r^2x = r^2tr^2x$ for some $t \in R$.

**Proof:**

Let $x$ be an element of $M$ and $r$ be an element of $R$. Since $r^2x \in r^2M$ and $r^2x \in <r^2x>$ implies $r^2x \in r^2M \cap <r^2x>$. But $M$ is 2-regular, then $r^2M \cap <r^2x> = r^2 <r^2x>$. Thus, $r^2x \in r^2<r^2x>$ implies $r^2x = r^2t r^2x$ for some $t \in R$.

**Proposition (1.4):**

Let $M$ be a module over principal ideal ring $R$. If for every element $x$ of $M$ and every element $r \in R$, $r^2x = r^2tr^2x$ for some $t \in R$ implies $M$ is a 2-regular module.

**Proof:**

Let $N$ be a submodule of $M$ and $I$ is an ideal of $R$. First, to prove $r^2M \cap N = r^2N$ for every element $r \in R$. Let $x \in r^2M \cap N$ implies $x \in r^2M$, $x \in N$. Thus $x = r^2m$ for some $m \in M$. Then $x = r^2tr^2m$ for some $t \in R$ by hypothesis. Hence $x \in r^2N$. But $R$ is a principal ideal ring. Therefore $r^2M \cap N = r^2N$.

**Proposition (1.5):**

Let $M$ be a cyclic $R$-module. If for every element $x$ of $M$ and every element $r$ of $R$, $r^2x = r^2tr^2x$ for some $t \in R$, implies $M$ is a 2-regular module.

**Proof:**

Let $M = Rm$ be a cyclic module for some $m \in M$. Let $N$ be a submodule of $M$ and $I$ is an ideal of $R$. Let $y \in I^2M \cap N$ then $y \in I^2M$ and $y \in N$. Thus $y = r^2m = r^2tr^2m \in r^2N$ for some $t \in R$ and $r \in I$. Therefore $y \in I^2N$ implies $M$ is 2-regular.

The proof of the following result is similar to that of propositions (1.3) and (1.4).

**Corollary (1.6):**

Let $R$ be a 2-regular ring then for every element $a \in R$, $a^2 = a^2t a^2$ for some $t \in R$, and the converse is true if $R$ is a principal ideal ring.

**Proposition (1.7):**

Let $R$ be a principal ideal ring and $M$ be an $R$-module. The following statements are equivalent:

1. $M$ is 2-regular module.
2. $\frac{R}{ann(x)}$ is 2-regular for every element $x$ of $M$.
3. For every element $x$ of $M$ and every element $r$ of $R$, $r^2x = r^2tr^2x$ for some $t \in R$.
Proof:
(1) \implies (3) It follows by Proposition (1.3).
(3) \implies (1) By Proposition (1.4).
(1) \implies (2) Let \( r + \frac{\text{ann}(x)}{R} \in \frac{R}{\text{ann}(x)} \) where \( x \in M \) and \( r \in R \).

Since \( M \) is 2-regular, then \( r^2x = r^2tr^2x \) for some \( t \in R \). Thus \( r^2 - r^2tr^2 \in \frac{\text{ann}(x)}{R} \) implies \( \frac{R}{\text{ann}(x)} \) is 2-regular.

(2) \implies (1) Let \( x \in M \) and \( r \in R \). Since \( \frac{R}{\text{ann}(x)} \) is 2-regular, then \( r^2 + \frac{\text{ann}(x)}{R} = (r^2 + \frac{\text{ann}(x)}{R}) (t + \frac{\text{ann}(x)}{R}) (r^2 + \frac{\text{ann}(x)}{R}) \) for some \( t \in R \). Thus \( r^2x = r^2tr^2x \) implies \( M \) is 2-regular.

We have the following results:

Corollary (1.8):
Let \( R \) be a principal ideal ring. Then \( R \) is 2-regular if and only if all \( R \)-modules are 2-regular.

Proof:
(\Rightarrow) Let \( R \) be 2-regular ring and \( M \) is an \( R \)-module. Then \( \frac{R}{\text{ann}(x)} \) is 2-regular for every element \( x \in M \) by [1,Cor.(3.3)]. Therefore \( M \) is 2-regular by proposition (1.7).

(\Leftarrow) Assume all \( R \)-modules are 2-regular. Thus \( R \) is 2-regular \( R \)-module. By Proposition (1.7), \( \frac{R}{\text{ann}(x)} \) is 2-regular for some every element \( x \in R \), so if take \( x = 1 \in R \) implies \( \frac{R}{\text{ann}(x)} = \frac{R}{\text{ann}(x)} \cong R \), therefore \( R \) is 2-regular.

Corollary (1.9):
Let \( R \) be a principal ideal ring. Then \( R \) is a 2-regular if and only if \( R \) is 2-regular \( R \)-module.

Proof: By the same argument of Corollary (1.8).

Corollary (1.10):
Let \( R \) be a principal ideal ring. If \( \frac{R}{\text{ann}(M)} \) is 2-regular then \( M \) is 2-regular \( R \)-module.

Proof: Let \( x \) be a non-zero element of \( M \). Since \( \text{ann}(M) \subseteq \frac{\text{ann}(x)}{R} \), there exists an epimorphism \( f: \frac{R}{\text{ann}(M)} \to \frac{R}{\text{ann}(x)} \) defined by \( f(r + \frac{\text{ann}(M)}{R}) = r + \frac{\text{ann}(x)}{R} \). Therefore \( \frac{R}{\text{ann}(x)} \) is 2-regular by [1,Cor.(3.3)]. Then \( M \) is 2-regular by Proposition (1.7).
2- Regular Modules and Other Related Modules

In this section, we study the relationships between 2-regular modules and other modules such as semiprime, divisible, projective and multiplication modules.

Recall that a proper submodule $N$ of an $R$-module $M$ is called a semiprime submodule if for every $r \in R, x \in M, k \in \mathbb{Z}^+$ such that $rkx \in N$ implies $rx \in N$ implies $rx \in N$, see [4]. Equivalently, a proper submodule $N$ of $M$ is semiprime if for every $r \in R, x \in M$ such that $r^2x \in N$ implies $rx \in N$, see [5].

An $R$-module $M$ is called semiprime if $<0>$ is a semiprime submodule of $M$.

The proof of the following result follows by [5].

**Proposition (2.1):**

Let $R$ be a principal ideal ring and $M$ is an $R$-module. If every proper submodule of $M$ is semiprime then $M$ is a 2-regular module. The converse is not true, for example: The module $\mathbb{Z}^4$ as $\mathbb{Z}$-module is 2-regular but $<0>$ is not semiprime.

The following proposition gives a partial converse of proposition (2.1).

**Proposition (2.2):**

Let $M$ be 2-regular and semiprime $R$-module then every proper submodule of $M$ is semiprime.

**Proof:**

Let $N$ be a proper submodule of $M$ and $r^2x \in N$ where $r \in R, x \in M$ implies $r^2x \in r^2M \cap N = r^2N$ since $M$ is 2-regular. Then $r^2x = r^2n$ for some $n \in N$, thus $r^2(x - n) \in <0>$. But $<0>$ is semiprime, hence $rx = rn \in N$. Therefore $N$ is semiprime submodule of $M$.

Before we give a consequence of Proposition (2.2), we need the following lemma:

**Lemma (2.3):**

Let $M$ be 2-regular and semiprime $R$-module then $J(R)M = <0>$.

**Proof:**

Let $r \in J(R)$ and $x \in M$ then $r^2x = r^2t^2x$ for some $t \in R$ since $M$ is 2-regular, $r^2x(1 - r^2t) = 0$ implies $1 - r^2t$ is invertible in $R$. Then $r^2x = 0$, but $M$ is semiprime thus $rx = 0$. Therefore $J(R)M = <0>$.

Recall that an $R$-module $M$ is called semisimple if every submodule of $M$ is a summand. The sum of all simple submodules of a module $M$ is called the socle of $M$ is denoted by $\text{Soc}(M)$, moreover if $\text{Soc}(M) = 0$, then $M$ has no simple submodule and if $\text{Soc}(M) = M$ then $M$ is semisimple module, see [6].

A commutative ring is a local ring in case it has a unique maximal ideal, see [7].

**Corollary (2.4):**

Let $R$ be a local ring and $M$ is 2-regular and semiprime $R$-module then $M$ is a semisimple and hence is regular.
Proof:

Since $R$ is a local ring, then $\frac{R}{J(R)}$ is a simple ring and hence is semisimple. By [6], $\text{Soc}(M) = \text{ann}(J(R)) = \{m \in M; mJ(R) = 0\}$. But $J(R)M = \langle 0 \rangle$ by lemma (2.3), thus $\text{Soc}(M) = M$. Therefore $M$ is semisimple.

Now, we have the following:

**Proposition (2.5):**

Let $N$ be a semiprime submodule of an $R$-module $M$ and $K$ is a 2-pure submodule of $M$ containing $N$, then $\frac{K}{N}$ is semiprime submodule in $\frac{M}{N}$.

**Proof:**

Let $r^2(x + N) \in \frac{K}{N}$ for some $r \in R$ and $x + N \in \frac{M}{N}$. Then $r^2x \in K$, implies $r^2x \in r^2M \cap K = r^2K$ since $K$ is 2-pure in $M$. Let $r^2x = r^2m$ for some $m \in K$. Thus $r^2(x - m) = 0 \in N$ implies $r(x - m) \in N$ since $N$ is semiprime submodule in $M$, hence $r(x + N) = rm + N \in \frac{K}{N}$. Therefore $\frac{K}{N}$ is semiprime submodule in $\frac{M}{N}$.

**Corollary (2.6):**

Let $N$ be a semiprime submodule of an $R$-module $M$ and $K$ is a 2-pure in $M$ with $N \subseteq K$ then $K$ is semiprime submodule in $M$.

**Proof:**

Let $r^2x \in K$ for some $r \in R$ and $x \in M$. Thus $r^2(x + N) \in \frac{K}{N}$, but $\frac{K}{N}$ is semiprime in $\frac{M}{K}$ by Proposition (2.5) therefore $r(x + N) \in \frac{K}{N}$. Hence $rx \in K$, that is $K$ is semiprime in $M$.

Let $R$ be an integral domain, an $R$-module $M$ is said to be divisible if and only if $rM = M$ for every non-zero element $r$ of $R$, see [8].

An $R$-module $M$ is said to be a prime module if $\text{ann}(M) = \text{ann}(N)$ for every non-zero submodule $N$ of $M$, see [9].

**Proposition (2.7):**

Let $M$ be a module over a principal ideal domain $R$ and $N$ is a divisible $R$-submodule of $M$ then $N$ is a 2-pure submodule in $M$.

**Proof:** Since $N$ is divisible then for each $r \in R$, $r^2N = N$. Therefore $N \cap r^2M = r^2N$. 

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Remark (2.8):
The converse of proposition (2.7) is not true, for example: the submodule \( \{0, 2\} \) of the module \( \mathbb{Z}_4 \) as \( \mathbb{Z} \)-module where \( \{0, 2\} \) is 2-pure in \( \mathbb{Z}_4 \), but is not divisible since there exists \( 2 \in \mathbb{Z} \) and \( 2 \cdot \{0, 2\} = \{0\} \). That is \( 2 \cdot \{0, 2\} \neq \{0, 2\} \).

The following proposition gives a condition under which the converse of proposition (2.7) is true.

**Proposition (2.9):**
Let \( M \) be divisible module over a principal ideal domain \( R \) and \( N \) is a 2-pure in \( M \) then \( N \) is divisible.

**Proof:**
Assume \( N \) is 2-pure in \( M \), let \( m \in N \) and \( r \in R \). Since \( M \) is divisible implies \( m = r^2x \) for some \( x \in M \). But \( m = r^2x \in r^2M \cap N = r^2N \subseteq rN \). Therefore \( N = rN \).

As an immediate consequence we have the following:

**Corollary (2.10):**
Let \( R \) be a principal ideal domain and every proper submodule of an \( R \)-module \( M \) is divisible then \( M \) is 2-regular. The converse is true if \( M \) is divisible.

**Proof:**
Follows by Propositions (2.7) and (2.9).

**Corollary (2.11):**
Let \( R \) be a principal ideal domain and \( M \) is 2-regular and divisible \( R \)-module then \( M \) is prime module.

**Proof:**
By above corollary (2.10), every submodule \( N \) of \( M \) is divisible. Thus \( rN = N \) for every \( r \in R \). Therefore \( \text{ann}(N) = \text{ann}(M) = \{0\} \). Hence \( M \) is prime module.

**Corollary (2.12):**
Let \( R \) be a principal ideal domain and \( M \) is 2-regular injective \( R \)-module then \( M \) is prime module.

**Proof:** Clear

We give the following theorem.

**Theorem (2.13):**
Let \( R \) be any ring. The following statements are equivalent:

1. \( \bigoplus \lambda A \) is 2-regular \( R \)-module for any index set \( \Lambda \).
2. Every projective \( R \)-module is 2-regulaar module.

**Proof:**
Let $M$ be projective $R$-module then there exists a free $R$-module $F$ and an $R$-epimorphism $f : F \rightarrow M$, and $F \cong \bigoplus \Lambda R$ where $\Lambda$ is an index set. We have the following short exact sequence

$$0 \rightarrow \ker f \xrightarrow{i} \bigoplus \Lambda R \xrightarrow{f} M \rightarrow 0$$

Where $i$ is the inclusion mapping.

Since $M$ is projective, the sequence is split implies that $\bigoplus \Lambda R \cong \ker f \oplus M$. But $\bigoplus \Lambda R$ is 2-regular $R$-module. Therefore by [1,Cor.(3.4)] $M$ is 2-regular module.

Assume that every projective $R$-module is 2-regular module. Since $R$ is projective $R$-module, then $\bigoplus \Lambda R$ is projective because the direct sum of projective modules is projective. Therefore $\bigoplus \Lambda R$ is 2-regular $R$-module for any index set $\Lambda$.

Recall that an $R$-module $M$ is called multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$, see [10] We have the following:

**Proposition (2.14):**

If $M$ is a finitely generated faithful multiplication $R$-module. The following statements are equivalent:

1. $R$ is 2-regular ring.
2. $M$ is 2-regular $R$-module.

**Proof:**

(1) $\Rightarrow$ (2) Let $N$ be a submodule of $M$ and $I$ is an ideal of $R$. Since

$$I^2 M \cap N = I^2 M \cap JM$$

for some ideal $J$ of $R$

$$= (I^2 \cap J)M$$

since $M$ is faithful multiplication, see [10]

$$= I^2 JM$$

since $R$ is 2-regular

$$= I^2 (JM)$$

Thus $I^2 \cap J = I^2 J$ since $M$ is finitely generated faithful multiplication, see [10]. Therefore $R$ is 2-regular ring.

Recall that an $R$-module $M$ is said to be $I$-multiplication module if each submodule $N$ of $M$ of the form $JM$ for some idempotent ideal $J$ of $R$, see [11]. It is clear that every $I$-multiplication module is multiplication but not the converse. Clearly the two concepts multiplication and $I$-multiplication modules are equivalent over regular rings. However we have the following:

**Proposition (2.15):**

If $M$ is $I$-multiplication and 2-regular $R$-module then $M$ is regular module.

**Proof:**

Let $N$ be a submodule of $M$ and $I$ is an ideal of $R$. Since
\[ IM \cap N = IM \cap JM \]
\[ = IM \cap J^2M \quad \text{for some idempotent } J = J^2 \]
\[ = J^2(IM) \quad \text{since } M \text{ is 2-regular} \]
\[ = (I^2J)M \quad \text{since } R \text{ is 2-regular} \]
\[ = I(J^2M) \]
\[ = I(JM) \]
\[ = IN \]

Therefore \( M \) is regular module.

**Proposition (2.16):**

If \( M \) is I-multiplication and 2-regular \( R \)-module then every submodule \( N \) of \( M \) is I-multiplication as \( R \)-module.

**Proof:**

Let \( N \) be a submodule of \( M \) and \( K \) is any submodule in \( N \), then \( K \) is a submodule of \( M \) and \( K = IM = I^2M \) for some idempotent ideal \( I \) of \( R \). Since

\[ K = N \cap K \]
\[ = N \cap I^2M \]
\[ = I^2N \quad \text{because } M \text{ is 2-regular} \]
\[ = IN \]

Thus \( N \) is I-multiplication \( R \)-module.

**References**

المقاسات المنتظمة من النمط - 2

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الخلاصة

نظراً مقياس M بأنه منظم من النمط - 2 إذا كان كل مقياس R فإنه مقياس E على R إذا كانت جزئي من النمط-2 ، إذا غالب عن المقاس الجزئي N بأنه نقي من النمط-2 في M إذا حقق M\\cap N = I لكل مثالي I في R. [1].

في هذا البحث نستعمل دراسة مفهوم النتظام من النمط-2 [1]. في القسم الأول من هذا البحث أظهرنا تميزة للمقاسات المنتظمة من النمط-2 . في القسم الثاني درسنا العلاقة بين المقاسات المنتظمة من النمط-2 وايضاً أخر من المقاسات.

الكلمات المفتاحية : المقاسات الجزئية النقي من النمط - 2 ، المقاسات المنتظمة من النمط - 2 ، المقاسات الجزئية النقي،
المقاسات المنتظمة.